MTH 201: Multivariable Calculus and Differential Equations

Homework II

(Due 02/09)

1. For each of the following scalar fields, compute all first-order partial derivatives. Also, verify that $D_1(D_2 f) = D_2(D_1 f)$.

   (a) $f(x, y) = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$.

   (b) $f(x, y) = \log(x^2 + y^2)$.

2. Let f be a scalar field.

   (a) If $f'(x; y) = 0$ for every x in some n-ball $B(a)$ and for every vector y, then prove that f is constant on $B(a)$. [Hint: Use the Mean-Value Theorem]

   (b) If $f'(x; y) = 0$ for fixed vector y and for every vector x in some n-ball $B(a)$, then what can we say about f?

   (c) For a fixed vector a, prove that $f'(a; y) > 0$ cannot hold good for every nonzero vector y.

3. A set S in $\mathbb{R}^n$ is called *convex* if for every pair of points a and b in S, the line segment from a to b is also in S; in other words, $ta + (1-t)b \in S$ for each $t \in [0, 1]$.

   (a) Prove that every n-ball is convex.

   (b) If $f'(x; y) = 0$ for every x in an open convex set S and for every $y \in \mathbb{R}^n$, prove that f is constant on S.

4. Evaluate the directional derivatives of the following scalar fields for the points and directions given:

   (a) $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at $(1, 1, 0)$ in the direction of $i - j + 2k$.

   (b) $f(x, y, z) = (x/y)^z$ at $(1, 1, 1)$ in the direction of $2i + j - k$.

   (c) $f(x, y, z) = 3x - 5y + 2z$ at $(2, 2, 1)$ in direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 9$.

   (d) $f(x, y, z) = x^2 - y^2$ at $(3, 4, 5)$ along the curve of intersection of $2x^2 + 2y^2 - z^2 = 25$ and $x^2 + y^2 = z^2$.

5. Let f and g are scalar fields that are differentiable on an open set S. Derive the following properties of the gradient.

   (a) $\nabla(fg) = f\nabla g + g\nabla f$.

   (b) $\nabla(\frac{f}{g}) = \dfrac{g\nabla f - f\nabla g}{g^2}$, where $g \neq 0$.

6. Assume that f is differentiable at each point of an n-ball $B(a)$.

   (a) If $\nabla f(x) = 0$ for every $x \in B(a)$, prove that f is constant on $B(a)$.

   (b) If $f(x) \leq f(a)$ for every $x \in B(a)$, prove that $\nabla f(a) = 0$.

7. Find a equation of the tangent plane to the surface $xyz = a^3$ at a general point (x_0, y_0, z_0). Show that the volume of the tetrahedron bounded by this plane and the three coordinate planes is $9a^3/2$.

8. Find a constant c such that at any point of the intersection of the two spheres

$$(x - c)^2 + y^2 + z^2 = 3 \text{ and } x^2 + (y - 1)^2 + z^2 = 1,$$

the corresponding tangent planes will be perpendicular to each other.

9. If r_1 and r_2 denote the distances from a point (x, y) on the ellipse to its foci, show that the equation $r_1 + r_2 =$ constant (satisfied by these distances) implies the relation

$$T \cdot \nabla(r_1 + r_2) = 0,$$

where T is the unit tangent to the curve.